

# Dynamical relativistic corrections to the leptonic decay width of heavy quarkonia

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**Abstract.** We calculate the dynamical relativistic corrections, originating from radiative one-gluon exchange, to the leptonic decay width of heavy quarkonia in the framework of a covariant formulation of light-front dynamics. Comparison with the non-relativistic calculations of the leptonic decay width of  $J = 1$  charmonium and bottomonium  $S$ -ground states shows that relativistic corrections are large. Most importantly, the calculation of these dynamical relativistic corrections legitimate a perturbative expansion in  $\alpha_s$ , even in the charmonium sector. This is in contrast with the ongoing belief based on calculations in the non-relativistic limit. Consequences for the ability of several phenomenological potentials to describe these decays are described.

## 1 Introduction

Although the structure of heavy quarkonia in terms of a heavy quark–antiquark non-relativistic bound state has been known for a long time, many recent developments show the importance of a relativistic description of such states. Mainly two paths are followed.

- (1) One can first estimate relativistic corrections to the  $\bar{q}q$  bound state in a  $v/c$  expansion. This is the aim of the NRQCD formalism [1]. When the quark mass is large enough, this is certainly adequate since this expansion should converge rapidly. It is however not clear whether the charm quark mass is heavy enough for such an expansion to be valid. To check it out, one should compare the results of NRQCD with relativistic calculations.
- (2) Relativistic corrections can also be calculated in a relativistic framework to describe the two-body bound state, such as the Bethe–Salpeter formalism, or light-front dynamics (LFD). In a previous article [2], we have investigated the relativistic kinematical corrections to the leptonic decay width of the  $J/\psi$  and the  $\Upsilon$  induced by the finite momentum between their quark and antiquark. We used an appropriate formalism, the so-called “covariant formulation of light-front dynamics” (CLFD), the details of which can be found in [3]. As pointed out in this reference, the CLFD allows a straightforward comparison between relativistic and non-relativistic calculations.

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We shall investigate in this article the importance of dynamical relativistic corrections. Kinematical relativistic corrections can be calculated without any knowledge of the dynamical origin of the two-body wave function. Dynamical corrections, on the other hand, correspond to relativistic corrections to the wave function itself. While the non-relativistic two-body wave function of the  $J = 1$  state has only two components (the  $S$  and  $D$  states), the relativistic wave function formulated on the light-front has six dynamical components [3]. If the dynamical origin of the two-body interaction is known, these components can, in principle, be calculated exactly or else calculated in perturbation theory, starting from the non-relativistic components. This has been done for instance in [4], for the calculation of the relativistic corrections to the deuteron wave function.

In the case of heavy quarkonium, the two body interaction has two main parts: a confining interaction, and the interaction coming from the exchange of an effective gluon (one-gluon exchange, OGE). The latter interaction gives rise, in the non-relativistic limit, to the well-known Coulomb interaction. While little is known of the dynamical origin of the confining potential, one may hope that for heavy quarkonia, the two-body wave function is mostly sensitive, for relativistic corrections, to the short range part of the interaction, i.e. to the one-gluon exchange potential and its  $1/r$  non-relativistic behaviour.

This is the usual assumption made when calculating the dynamical relativistic correction coming from the OGE interaction. In zeroth order in a  $\vec{k}^2/m^2$  expansion, where  $\vec{k}$  is the relative momentum of the quark–antiquark pair in the two-body bound state, it gives rise to the fol-

lowing well-known correction to the leptonic decay width of  $J = 1$  states [5,6]:

$$\Gamma_1^{\text{NR}} = \Gamma_0^{\text{NR}} \left( 1 - \frac{16\alpha_s}{3\pi} \right). \quad (1)$$

In the case of the  $J/\Psi$  leptonic decay width, this contribution amounts to a 50% reduction. This large correction clearly calls for a relativistic calculation of this contribution to all orders in  $\vec{k}^2/m^2$ . This is the aim of the present article.

We shall follow very closely the procedure of [2]. We refer the reader to this reference, and to [3] for a review article on CLFD. We introduce in Sect. 2 the scheme of the calculation of the leptonic decay width. The dynamical relativistic corrections are introduced in Sect. 3. Numerical results are presented in Sect. 4, and our conclusions and perspectives are drawn out in Sect. 5.

## 2 The leptonic decay width of heavy quarkonium

### 2.1 The quarkonium wave function in CLFD

Among the various ways to deal with relativity in the description of bound states, we will focus on LFD. In the standard formulation of LFD [7], the wave function of a bound system is defined on a plane characterised by the equation  $t + z/c = 0$ . The usual Schrödinger, equal time, formalism is easily recovered in LFD by letting  $c$  go to infinity.

The description of relativistic heavy quark systems in LFD has many advantages, of which the most important one may be the absence of vacuum fluctuations. Accordingly, a meaningful decomposition of the state vector describing the system in terms of Fock components of definite number of particles is possible. Of course, the number of Fock components to be considered in any practical calculation depends on the dynamics of the system, and on the kinematical regime one is interested in.

The most serious drawback of this formulation, however, is that the position of the light-front  $t + z = 0$  (with  $c = 1$ ) is not rotational invariant. Since rotations in the  $zx$  and  $zy$  planes change the position of the light-front, the associated generators shall depend on the dynamics and cannot be reduced to kinematical transformations [8]. This means that one needs to know the complete dynamics in order to write down the general structure of a bound state of definite angular momentum. It also means that any electromagnetic operator should have the same (dynamical) transformation properties to match the bound state wave function one's. This is essential to guarantee that any physical amplitude (or cross-section) does not depend on the particular choice of the light-front we start with.

We therefore need an explicit procedure to exhibit in a convenient way these dynamical transformations. This is achieved in the covariant formulation of LFD. Our starting point is the invariant definition of the light-front by

$\omega \cdot x = 0$  where  $\omega$  is a (unspecified) light-like four vector ( $\omega^2 = 0$ ). The standard formulation of LFD can be easily recovered with the particular choice  $\omega = (1, 0, 0, -1)$  for the light-like four vector.

This definition of the light-front is explicitly invariant by any four-dimensional rotations, or any three-dimensional rotations and Lorentz boosts. Consequently, these transformations become  $\omega$ -dependent, but do not necessitate the knowledge of the dynamics of the system to construct them explicitly. The dynamics enters now into the  $\omega$ -dependence of the wave function and of the electromagnetic operator, in such a way that any physical amplitude should not depend on the particular position of the light-front, i.e. on  $\omega$ , unless approximations have been made. In this case, which is almost always true in practice, the explicit covariance of the approach enables us to exhibit the  $\omega$ -dependence of the amplitude and to separate the physical part from the non-physical,  $\omega$ -dependent one, as we shall explain below for the leptonic decay amplitude.

The wave function  $\Phi$ , of a two-body bound state, can be decomposed in terms of all the possible independent spin structures compatible with the quantum numbers of the state studied. In the particular case of vector mesons, which we are interested in, we can write

$$\Phi_{\sigma_2\sigma_1}^\lambda(k_1, k_2, p, \omega) = \sqrt{m} e_\mu^\lambda(p) \bar{u}^{\sigma_2}(k_2) \phi^\mu v^{\sigma_1}(k_1), \quad (2)$$

with

$$\begin{aligned} \phi^\mu = & \varphi_1 \frac{(k_1 - k_2)^\mu}{2m^2} + \varphi_2 \frac{1}{m} \gamma^\mu + \varphi_3 \frac{\omega^\mu}{\omega \cdot p} + \varphi_4 \frac{(k_1 - k_2)^\mu \not{\omega}}{2m\omega \cdot p} \\ & + \varphi_5 \frac{i}{2m^2\omega \cdot p} \gamma_5 \epsilon^{\mu\nu\rho\gamma} (k_1 + k_2)_\nu (k_1 - k_2)_\rho \omega_\gamma \\ & + \varphi_6 \frac{m\omega^\mu \not{\omega}}{(\omega \cdot p)^2}. \end{aligned} \quad (3)$$

This decomposition is similar to the decomposition of the deuteron wave function found in [4]. The six components of the wave function,  $\varphi_1$ - $\varphi_6$ , depend on two invariants. In order to make a close connection to the non-relativistic case, it will be convenient to introduce another pair of variables [3] defined by

$$\vec{k} = L^{-1}(\mathcal{P})\vec{k}_1 = \vec{k}_1 - \frac{\vec{\mathcal{P}}}{\sqrt{\mathcal{P}^2}} \left[ k_{10} - \frac{\vec{k}_1 \cdot \vec{\mathcal{P}}}{\sqrt{\mathcal{P}^2 + \mathcal{P}_0}} \right], \quad (4)$$

$$\vec{n} = \frac{L^{-1}(\mathcal{P})\vec{\omega}}{|L^{-1}(\mathcal{P})\vec{\omega}|} = \sqrt{\mathcal{P}^2} \frac{L^{-1}(\mathcal{P})\vec{\omega}}{\omega \cdot p}, \quad (5)$$

where

$$\mathcal{P} = k_1 + k_2. \quad (6)$$

The relativistic momentum  $\vec{k}$  corresponds, in the frame where  $\vec{k}_1 + \vec{k}_2 = \vec{0}$ , to the usual relative momentum between the two particles. Note that this choice of variable does not assume that we restrict ourselves to this particular frame. The unit vector  $\vec{n}$  corresponds, in this frame, to the spatial direction of  $\vec{\omega}$ . In terms of these variables, the wave function takes a form similar to the non-relativistic one:

$$\Psi_{\sigma_2\sigma_1}^\lambda(\vec{k}, \vec{n}) = \sqrt{m} w_{\sigma_2}^\dagger \psi^\lambda(\vec{k}, \vec{n}) w_{\sigma_1}, \quad (7)$$

with

$$\begin{aligned} \vec{\psi}(\vec{k}, \vec{n}) = & f_1 \frac{1}{\sqrt{2}} \vec{\sigma} + f_2 \frac{1}{2} \left( \frac{3\vec{k}(\vec{k} \cdot \vec{\sigma})}{\vec{k}^2} - \vec{\sigma} \right) \\ & + f_3 \frac{1}{2} (3\vec{n}(\vec{n} \cdot \vec{\sigma}) - \vec{\sigma}) \\ & + f_4 \frac{1}{2k} \left( 3\vec{k}(\vec{n} \cdot \vec{\sigma}) + 3\vec{n}(\vec{k} \cdot \vec{\sigma}) - 2(\vec{k} \cdot \vec{n})\vec{\sigma} \right) \\ & + f_5 \sqrt{\frac{3}{2}} \frac{i}{k} [\vec{k} \times \vec{n}] + f_6 \frac{\sqrt{3}}{2k} [[\vec{k} \times \vec{n}] \times \vec{\sigma}], \end{aligned} \quad (8)$$

where  $w_\sigma$  is the two-component Pauli spinor normalised to  $w_\sigma^\dagger w_\sigma = 1$ , and  $\vec{\sigma}$  are the usual Pauli matrices. The relation between  $\psi^\lambda$  and  $\vec{\psi}$ , is the same as the relation between the spherical function  $Y_1^\lambda(\vec{n})$  and  $\vec{n}$ .

The coefficients of the spin structures in (3) and (8) are scalar functions of two independent invariants, which we can choose as  $\vec{k}^2$  and  $\vec{k} \cdot \vec{n}$ , since these variables are only rotated by a Lorentz boost [3]. In the non-relativistic limit, only two components remain,  $f_1$  and  $f_2$ , and these only depend on  $\vec{k}^2$ . This can easily be seen if we keep track of the  $c$  factors, and then let  $c$  go to infinity to get the non-relativistic limit. In this study of heavy quarkonium states, one may neglect the tensor component,  $f_2$ , so one is left with the non-relativistic wave function  $f_1 \equiv \phi^{\text{NR}}(\vec{k}^2)$ . The relation between  $\varphi_1$ ,  $\varphi_2$  and  $\phi^{\text{NR}}$  is given by [3]

$$\varphi_1(\vec{k}^2) = \frac{m^2}{4\varepsilon_k(\varepsilon_k + m)} \sqrt{2} \phi^{\text{NR}}(\vec{k}^2), \quad (9)$$

$$\varphi_2(\vec{k}^2) = \frac{m}{4\varepsilon_k} \sqrt{2} \phi^{\text{NR}}(\vec{k}^2), \quad (10)$$

where  $\varepsilon_k = (\vec{k}^2 + m^2)^{1/2}$ . Note that the wave function  $f_1 \equiv \phi^{\text{NR}}$  is normalised according to

$$\int \left| \phi^{\text{NR}}(\vec{k}^2) \right|^2 \frac{d^3k}{(2\pi)^3} \frac{m}{\varepsilon_k} = 1. \quad (11)$$

## 2.2 The leptonic decay width

Since the total physical amplitude for the process  $M_J \rightarrow e^+e^-$  can be factorised into two separate amplitudes  $M_J \rightarrow \gamma$  and  $\gamma \rightarrow e^+e^-$ , the relevant physical information is completely included in the amplitude  $\mathcal{M}^{\mu\rho}$  to produce a photon with polarisation  $\epsilon^\mu$  from a vector quarkonium state of polarisation  $\epsilon^\rho$ .

Since our formulation of LFD is explicitly covariant, we can decompose  $\mathcal{M}^{\mu\rho}$  in terms of all possible tensor structures build up with the two four-momenta at our disposal,  $p$  and  $\omega$ , where  $p$  is the four-momentum of the quarkonium [2]. Thus, we can write

$$\begin{aligned} \mathcal{M}^{\mu\rho} = & F a_1^{\mu\rho} + \frac{B_1}{2\omega \cdot p} a_2^{\mu\rho} + \frac{B_2}{2\omega \cdot p} a_3^{\mu\rho} + B_3 \frac{M^2}{(\omega \cdot p)^2} a_4^{\mu\rho} \\ & + D a_5^{\mu\rho}, \end{aligned} \quad (12)$$

with

$$a_1^{\mu\rho} = g^{\mu\rho} - \frac{p^\mu p^\rho}{M^2}, \quad (13a)$$

$$a_2^{\mu\rho} = p^\mu \omega^\rho + p^\rho \omega^\mu, \quad (13b)$$

$$a_3^{\mu\rho} = p^\mu \omega^\rho - p^\rho \omega^\mu, \quad (13c)$$

$$a_4^{\mu\rho} = \omega^\mu \omega^\rho, \quad (13d)$$

$$a_5^{\mu\rho} = \frac{p^\mu p^\rho}{M^2}. \quad (13e)$$

We have denoted by  $M$  the mass of the quarkonium. Note that the term proportional to  $D$  does not contribute to the leptonic decay width. The amplitude  $\mathcal{M}^{\mu\rho}$  has two kinds of terms. The first one, proportional to  $F$ , is the physical contribution to the decay width. The other three, proportional to  $B_1$ ,  $B_2$  and  $B_3$ , are  $\omega$ -dependent contributions. In an exact calculation, the coefficients  $B_1$ ,  $B_2$  and  $B_3$  should be zero since the physical leptonic decay width should not depend on the particular orientation of the light-front one starts with. However, in any approximate calculation, these terms may be non-zero, but they are not physical. Consequently, they have to be eliminated in the computation of the physical leptonic width.

To extract  $F$  from the general amplitude  $\mathcal{M}^{\mu\rho}$ , one can first multiply  $\mathcal{M}^{\mu\rho}$  successively by the five tensor structures  $a_1$  to  $a_5$  given in (13). This gives a system of five coupled equations which is solved to get the physical amplitude  $F$ . Thus, we find

$$F = \frac{1}{2} (I_1 - 2I_2 + I_4 + I_5), \quad (14)$$

with

$$I_i = \mathcal{M}_{\mu\rho} a_i^{\mu\rho}. \quad (15)$$

These quantities are easily evaluated once  $\mathcal{M}^{\mu\rho}$  is calculated from the process under consideration, using the diagrammatical rules given in [3]. Given  $F$ , the decay width can be calculated [2], and is given by

$$\Gamma = \frac{4\pi}{3M^3} \alpha^2 e_q^2 |F|^2, \quad (16)$$

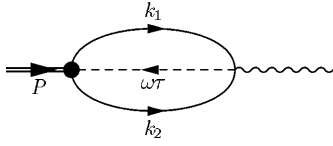
where  $e_q$  is the electric charge of the quark and  $\alpha$  is the electromagnetic fine structure constant.

## 2.3 The zeroth order calculation

The leading order contribution to the leptonic decay is shown in Fig. 1, where we have removed for simplicity the trivial vertex  $\gamma \rightarrow e^+e^-$ . Off-energy shell effects are governed by the variable  $\tau$ , which is unambiguously determined by four-momentum conservation and the on-mass shell condition for each particle [3]. Using the conservation law at the bound state vertex, we have<sup>1</sup>

$$k_1 + k_2 = p + \omega\tau. \quad (17)$$

<sup>1</sup> One should note that in CLFD, in the frame where we have  $\vec{k}_1 + \vec{k}_2 = \vec{0}$ , the total momentum  $\vec{p}$  is not  $\vec{0}$ . Instead  $\vec{p} + \vec{\omega}\tau$  is  $\vec{0}$ .



**Fig. 1.** Leading order contribution to the leptonic decay width. The dashed line represents the spurion (see text for details)

To keep track of this conservation law, we represent the four-vector  $\omega\tau$  by a dashed line in every diagram (the so-called spurion line, see [3] for more details). Note that the outgoing photon has virtuality  $q^2 = p^2$ . However, it can be assumed to be a “physical particle” in the sense that the final process  $\gamma \rightarrow e^+e^-$  is completely disconnected from the decay process  $M_J \rightarrow \gamma$ , and is exactly calculable in terms of the outgoing free electron and positron. As already explained in detail in [2], the zeroth order contribution to the amplitude, represented in Fig. 1, is given by

$$\mathcal{M}^{\mu\rho} = -\sqrt{3m} \int \frac{d^3k_1}{(2\pi)^3 2\varepsilon_{k_1} (1-x)} \times \text{Tr} [\gamma^\mu (\not{k}_1 + m) \phi^\rho (m - \not{k}_2)], \quad (18)$$

where  $x = \omega \cdot k_1 / \omega \cdot p$  and  $\phi^\rho$  is defined in (3). The physical part of the amplitude in zeroth order calculated by (14),  $F_0$ , can now be written in the form

$$F_0 = \int \frac{d^3k}{(2\pi)^3} \frac{m}{\varepsilon_k} O_0(\vec{k}^2) \phi^{\text{NR}}(\vec{k}^2), \quad (19)$$

where  $O_0(\vec{k}^2)$  is

$$O_0(\vec{k}^2) = -2\sqrt{6m} \left[ 1 + \frac{2m}{3\varepsilon_k} \left( 1 - \frac{\varepsilon_k}{m} \right)^2 \right]. \quad (20)$$

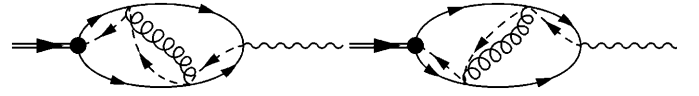
Note that in leading order in a  $\vec{k}^2/m^2$  expansion, kinematical relativistic corrections originate only from the factor  $m/\varepsilon_k$  of the relativistic phase space volume. In the non-relativistic limit, one has

$$F_0^{\text{NR}} = -2\sqrt{6m} \phi^{\text{NR}}(r=0). \quad (21)$$

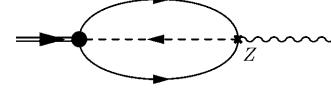
## 3 Dynamical relativistic corrections

### 3.1 Radiative corrections

As already pointed out, the dynamical relativistic corrections necessitate the knowledge of the dynamical origin of the two-body wave function, i.e., the way quark–antiquark states are bound. This goal is still far from being achieved. The standard assumption to overcome this issue is to suppose that for quark masses heavy enough, the dynamics is governed by a perturbative one-gluon exchange. The physical amplitude  $\mathcal{M}^{\mu\rho}$  for the relevant processes are indicated in Fig. 2. They can be calculated analogously to (18) using the diagrammatical rules given in [3]. The details of the calculation are too lengthy to be shown here,



**Fig. 2.** Radiative correction to the leptonic decay width



**Fig. 3.** Vertex renormalisation contribution to the leptonic decay width

but present no particular difficulties. The amplitude is both ultraviolet and infrared divergent. The computation of these contributions is given details of in the appendix.

Since these diagrams are ultraviolet divergent, we also need to include the renormalisation of the quark charge at the electromagnetic vertex, as has been already treated in detail in [9]. The corresponding diagram for this contribution is shown in Fig. 3. The counterterm  $Z$  is given by

$$Z = \frac{4}{3} \alpha_s \left[ \frac{9}{8\pi} + \frac{1}{2\pi} \log \left( \frac{\mu^2}{m^2} \right) + \frac{1}{4\pi} \log \left( \frac{\Lambda^2}{m^2} \right) \right], \quad (22)$$

where the leading  $4/3$  is a factor of colour generated by the gluon exchange.

The ultraviolet regularisation of the amplitude is done according to the Pauli–Villars prescription, with a cut-off  $\Lambda$ . This is the same prescription as in [9]. The amplitude is made infrared finite by giving a small non-zero mass  $\mu$  to the photon. We shall see in the next section that the final contribution remains infrared finite, as it should.

### 3.2 Coulomb subtraction

As is well known, one should be very careful in the evaluation of dynamical relativistic corrections as indicated in Fig. 2. Indeed, when we evaluate the zeroth order contribution shown in Fig. 1, with the non-relativistic wave function solution of the Schrödinger equation,  $\phi^{\text{NR}}(\vec{k}^2)$ , with the non-relativistic two-body potential

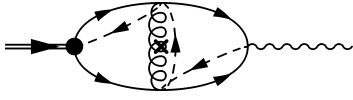
$$V_{\bar{q}q}(r) = V_{\text{OGE}}(r) + V_{\text{Conf}}(r), \quad (23)$$

we implicitly incorporate corrections of the type shown in Fig. 2, in leading non-relativistic order at least. One should therefore make sure that these contributions are properly removed from the relativistic calculation in order to avoid double counting.

The usual procedure to do this is to remove from the relativistic OGE the Coulomb interaction. We shall see below that our formulation of the leptonic decay width in CLFD enables us to have a clear handle on the contribution to remove, at any level of approximation.

Let us start from the Schrödinger equation, written in the form

$$\phi^{\text{NR}}(\vec{k}^2) = -\frac{4m}{s - M^2} \int \frac{d^3k'}{(2\pi)^3} V_{\bar{q}q}(\vec{k}'^2) \phi^{\text{NR}}(\vec{k}'^2), \quad (24)$$



**Fig. 4.** Coulomb contribution to the radiative correction of the leptonic decay width

with  $V_{\bar{q}q}$  given by (23), and  $s = 4(\vec{k}^2 + m^2)$ . The contribution to  $\phi^{\text{NR}}(\vec{k}^2)$  which corresponds in perturbation theory (first order in  $\alpha_s$ ) to the OGE interaction in the non-relativistic limit is therefore given by

$$\delta\phi^{\text{NR}}(\vec{k}^2) = \frac{4}{3} \frac{4m}{s - M^2} \int \frac{d^3k'}{(2\pi)^3} \frac{\alpha_s}{(\vec{k} - \vec{k}')^2} \phi^{\text{NR}}(\vec{k}'^2). \quad (25)$$

Inserted in the zeroth order calculation indicated in Fig. 1, and written in (19), it gives the contribution  $\delta F$  to the physical amplitude:

$$\delta F = \int \frac{d^3k}{(2\pi)^3} \frac{m}{\varepsilon_k} O_0(\vec{k}^2) \delta\phi^{\text{NR}}(\vec{k}^2). \quad (26)$$

This contribution is represented graphically in Fig. 4. It is ultraviolet finite, but diverges in the infrared region. We regularise the latter divergence by giving a small finite mass  $\mu$  to the gluon. Note also that in (25), the phase space volume in  $\vec{k}'$  is the non-relativistic one since it involves the non-relativistic wave function, the solution of the Schrödinger equation. However, in (26), it involves the relativistic phase space volume since one has to perform the integral over  $\vec{k}$  on the whole momentum range, as done in (19) for the calculation of kinematical relativistic corrections.

The contribution  $\delta F$  of (26), together with  $\delta\phi^{\text{NR}}(\vec{k}^2)$  given in (25), should be removed from the total relativistic amplitude  $\mathcal{M}^{\mu\rho}$  given by (18). In the non-relativistic limit for the two-body bound state (limit  $(\vec{k}'^2/m^2) \ll 1$ ), it gives the following contribution to  $F$ :

$$\delta F^{\text{NR}} = \frac{4}{3} \alpha_s \frac{m}{\mu} F_0^{\text{NR}}. \quad (27)$$

### 3.3 Non-relativistic limit

In the non-relativistic limit for the two-body bound state, the contribution of Fig. 2 to the physical amplitude is

$$F_{\text{OGE}}^{\text{NR}} = \frac{4}{3} \alpha_s \left[ \frac{m}{\mu} - \frac{7}{8\pi} + \frac{1}{2\pi} \log\left(\frac{\mu^2}{m^2}\right) + \frac{1}{4\pi} \log\left(\frac{\Lambda^2}{m^2}\right) \right] F_0^{\text{NR}}, \quad (28)$$

while the contribution of Fig. 3 gives, from (22),

$$F_Z^{\text{NR}} = -ZF_0^{\text{NR}} = -\frac{4}{3} \alpha_s \left[ \frac{9}{8\pi} + \frac{1}{2\pi} \log\left(\frac{\mu^2}{m^2}\right) + \frac{1}{4\pi} \log\left(\frac{\Lambda^2}{m^2}\right) \right] F_0^{\text{NR}}. \quad (29)$$

Adding the contributions of  $F_{\text{OGE}}^{\text{NR}}$  and  $F_Z^{\text{NR}}$ , then removing the contribution  $\delta F^{\text{NR}}$ , we get the final, infrared finite, contribution in the non-relativistic limit:

$$F_1^{\text{NR}} = -\frac{8}{3\pi} \alpha_s F_0^{\text{NR}}. \quad (30)$$

Added to  $F_0^{\text{NR}}$ , it gives the well-known radiative correction in leading order in  $\alpha_s$ , as indicated in (1) for the total width.

## 4 Numerical results

To easily compare our numerical calculations with the non-relativistic limit found above, we consider the following ratio:

$$R \equiv \frac{\Gamma_1}{\Gamma_0^{\text{NR}}} = \frac{\Gamma_1}{\Gamma_0} \frac{\Gamma_0}{\Gamma_0^{\text{NR}}}, \quad (31)$$

where  $\Gamma_1$  is the relativistic decay width to first order in  $\alpha_s$  (calculated in Sect. 3.1),  $\Gamma_0$  is the relativistic decay width in leading order in  $\alpha_s$  (calculated in Sect. 2.3), while  $\Gamma_0^{\text{NR}}$  is the non-relativistic limit in leading order in  $\alpha_s$  given by the Van Royen–Weisskopf formula:

$$\Gamma_0^{\text{NR}} = 16\pi e_q^2 \frac{\alpha^2}{M^2} |\phi^{\text{NR}}(r=0)|^2. \quad (32)$$

We also note

$$\frac{\Gamma_0}{\Gamma_0^{\text{NR}}} = R_K, \quad (33)$$

$$\frac{\Gamma_1}{\Gamma_0} = \left(1 - \frac{16\alpha_s}{3\pi} R_D\right). \quad (34)$$

The ratio  $R_K$  is simply the kinematical relativistic corrections we already calculated in [2]. The dynamical corrections have been written in the form of (34) to explicitly exhibit the non-relativistic limit of the radiative corrections when the two-body bound state wave function is restricted to very small momenta. In this case,  $R_D$  should go to unity.

For the hadronic vertex, we use two types of test wave functions to assess the importance of these relativistic corrections. On the one hand, we take an harmonic oscillator wave function corresponding to a confining potential for large distances between the quark and the antiquark. The wave function reads

$$\phi_{\text{H}}^{\text{NR}}(\vec{k}^2) = N \left(\frac{6\pi}{k_m^2}\right)^{3/4} \exp\left(-\frac{3}{4} \frac{\vec{k}^2}{k_m^2}\right). \quad (35)$$

In the previous expression,  $k_m^2$  is the mean relative momentum squared. In this case, the wave function at the origin can be written

$$\phi_{\text{H}}^{\text{NR}}(r=0) = N \left(\frac{2}{3\pi}\right)^{3/4} (k_m^2)^{3/4}. \quad (36)$$

On the other hand, we take the Coulomb wave function expected to dominate for very heavy quark masses. It is given by

$$\phi_C^{\text{NR}}(\vec{k}^2) = 8N\sqrt{\pi} \frac{(k_m^2)^{5/4}}{(\vec{k}^2 + k_m^2)^2}. \quad (37)$$

As in (35),  $k_m^2$  is the mean relative momentum squared. Thus, the wave function at the origin is

$$\phi_C^{\text{NR}}(r=0) = N \frac{1}{\sqrt{\pi}} (k_m^2)^{3/4}. \quad (38)$$

Note that the choice of these test wave functions is just a way to investigate the sensitivity of the decay constant to various shapes of the wave function. It is not intended to imply any assumption about the exact dynamics binding the heavy quark–antiquark pair. The normalisation factor  $N$  is calculated according to (11). It goes to 1 in the non-relativistic limit.

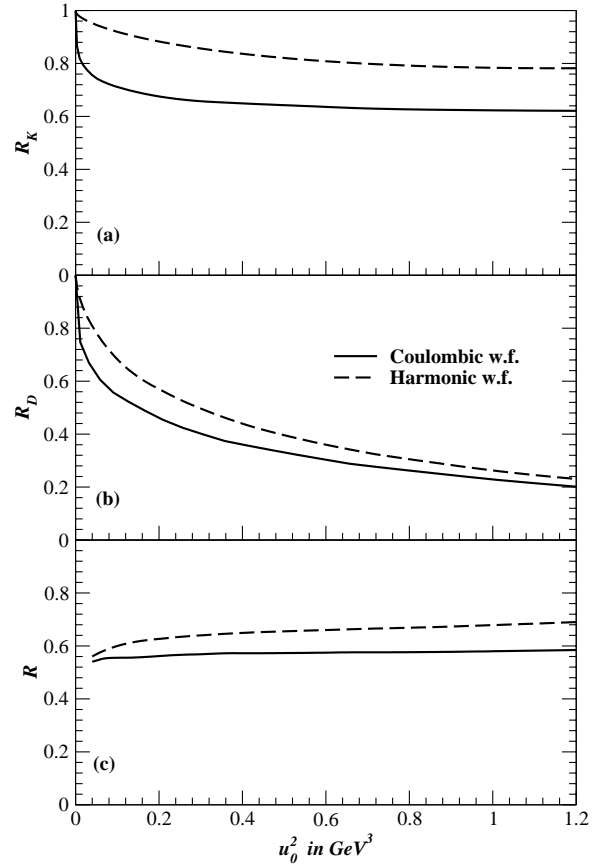
In Figs. 5 and 6, we plot our numerical results for the kinematical and dynamical (radiative) relativistic corrections, for the charmonium and the bottomonium ground state respectively, as a function of the square of the radial wave function at the origin,  $u_0$ , defined by

$$u_0 = \sqrt{4\pi}\phi(r=0). \quad (39)$$

This is indeed the relevant variable in this case since the leading, non-relativistic decay width,  $\Gamma_0^{\text{NR}}$ , as given in (32), is directly proportional to it. In the following numerical calculation, we choose  $\alpha_s = 0.3$  for the charmonium sector and  $\alpha_s = 0.15$  for the bottomonium sector.

The kinematical relativistic correction is shown in Fig. 5a for the  $J/\Psi$ , with  $m_c = M_{J/\Psi}/2$ , is the same as the one already calculated in [2] for three different phenomenological potentials. The results obtained here for our two kinds of test wave functions are very similar in shape. As expected, a larger reduction is obtained for the Coulombic wave function as it has larger high momentum components<sup>2</sup>. The dynamical relativistic corrections are indicated in Fig. 5b. They are much larger, with a sharp decrease as a function of  $u_0^2$ . Again, the Coulombic wave function has a larger correction. For the typical value of  $u_0^2$  of the order of  $1 \text{ GeV}^3$ , the correction is of the order of 0.25. This means that the well-known non-relativistic radiative correction,  $-(16\alpha_s)/(3\pi)$ , in (1) is reduced by a factor 1/4. This is consistent with the first estimate of this correction found in [10]. This result has an important and rather nice consequence: it justifies an expansion in  $\alpha_s$  in the charmonium sector, since the radiative correction is now of the order of 10–15%, as compared to the 50% reduction given by previous non-relativistic calculation with  $\alpha_s = 0.3$ .

The total correction to the decay width, represented by the ratio  $R$ , is shown in Fig. 5c. Surprisingly enough, this ratio is rather insensitive to the precise value of  $u_0^2$ , i.e., to the precise value of the radial wave function at the origin. Indeed, the two relativistic corrections (kinematical

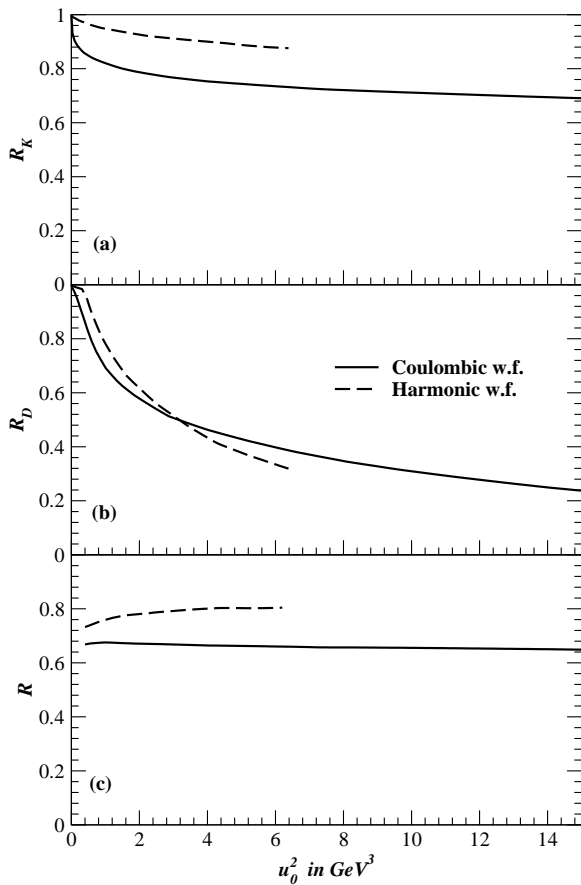


**Fig. 5a–c.** Relativistic corrections to the  $J/\Psi$  leptonic decay width, for the two test wave functions. We show both the Coulombic one (solid line) and the harmonic one (dashed line). These contributions are plotted as a function of  $u_0^2$ , defined in (39) and are proportional to the square of the wave function at the origin. In **a** we give the kinematical correction, ratio  $R_K$ , and **b** shows the dynamical correction, ratio  $R_D$ , and **c** the total correction, the ratio  $R$ . The mass of the charm quark is taken to be  $M_\Psi/2$

and dynamical) have opposite effects on  $R$  and tend to compensate each other, leading to an overall reduction factor varying from 0.6 (for a Coulombic wave function) to 0.7 (for an harmonic wave function). This indicates that to have an agreement with the new experimental decay width of the  $J/\Psi$  [11], the square of the radial wave function at the origin should be of the order of  $0.8\text{--}1 \text{ GeV}^3$ . This is the case for several potentials; see for example [12–15]. A pure Coulombic potential at short distance, like the Cornell potential [16], is however ruled out by our analysis, while a Coulombic potential modified according to asymptotic freedom is perfectly viable [12, 15].

The results for the  $\Upsilon$  state are indicated on Figs. 6. The behaviour is very similar to the one observed in the charmonium sector. The overall correction factor is of the order of 0.65 (for a Coulombic wave function) and 0.8 (for an harmonic wave function). The total correction factor is surprisingly large given the large bottom quark mass. However, the correction factor is much larger for a Coulombic wave function than an harmonic oscillator one.

<sup>2</sup> This correction even diverges in a naive  $\vec{k}^2/m^2$  expansion!



**Fig. 6a–c.** Same as in Fig. 5 but for the Upsilon decay width. The bottom quark mass is taken to be  $M_T/2$ . The values of  $u_0^2$  for the harmonic oscillator wave function have been deliberately restricted in order to correspond to physical ranges for the mean squared momentum between the quark–antiquark pair

This is indeed natural since for large quark mass the two-body wave function is concentrated to small distances. In this region, the high momentum components of the wave function play a major role in enhancing relativistic corrections.

In order to get agreement with the experimental decay width, the square of the wave function at the origin needs to be of the order of 6 to 8  $\text{GeV}^3$ . This has several interesting consequences. On the one hand, two potentials which are compatible with the  $J/\Psi$  decay width (logarithmic [14], and power law [13]) give a far too small decay width for the  $\Upsilon$  [17]. These potentials have no Coulombic components, and they lack high momentum components which show up in the  $\Upsilon$  sector. On the other hand, a pure Coulombic potential still gives a too large wave function at the origin. The two potentials of [12, 15], which correct the Coulomb interaction at short distances according to asymptotic freedom, are compatible with the experimental  $\Upsilon$  decay width<sup>3</sup>, as well as the  $J/\Psi$  one.

<sup>3</sup> The squared radial wave function at the origin is of the order of 6 to 6.5  $\text{GeV}^3$

## 5 Perspectives

We have presented a coherent formulation of the relativistic corrections to the leptonic decay width of heavy quarkonia. These corrections include both kinematical and dynamical contributions; the latter were calculated in first order in  $\alpha_s$ , but to all orders in  $\vec{k}^2/m^2$ .

As already found in our previous study [2], relativistic corrections are very large in the charmonium sector, and not negligible in the bottomonium sector. Kinematical relativistic corrections, which are independent of the dynamics binding the quark–antiquark pair, lead to a large reduction of the leptonic decay width. Dynamical relativistic corrections, which correspond to relativistic corrections to the two-body bound state itself, lead to a sizeable reduction of the standard correction  $16\alpha_s/3\pi$  found in the non-relativistic limit. This result is particularly important since it shows that an expansion in  $\alpha_s$  now becomes meaningful. Indeed, the first order correction in  $\alpha_s$  now amounts to about 10–15% correction in the case of the  $J/\Psi$ , down from 50% in the non-relativistic limit.

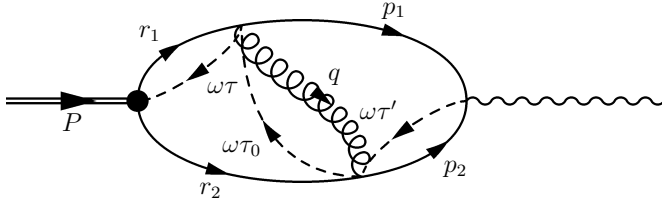
While the individual relativistic corrections (kinematical and dynamical) depend strongly on the mean momentum squared (or equivalently on the radial wave function at the origin), the total relativistic correction is remarkably constant as a function of the square of the wave function at the origin. The final correction is about 0.6 for a Coulombic wave function and 0.7 for an harmonic wave function, relative to the non-relativistic zeroth order calculation (Weisskopf–Van Royen approximation).

These relativistic corrections show that realistic phenomenological two-body potentials, such as the Richardson [12] or the Buchmüller–Tye [15] potentials, are able to reproduce the decay width of heavy quarkonia, both in the charmonium and the bottomonium sector. This solves a long-standing problem in this domain, as phenomenological two-body potentials were able to reproduce the whole spectrum of charmonium and bottomonium states, but not their leptonic decay width. As already explained in [2], the importance of relativistic corrections originates from the fact that the leptonic decay widths are very sensitive to the high momentum components of the wave function. This occurs because it involves the integral over the whole momentum range of the wave function and not the square of it, as in many other observables.

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## A Appendix

We give details in this appendix of the calculation of the radiative corrections to the leptonic decay width. Here, we concentrate on the first diagram shown in Fig. 2. The second one can be calculated analogously. The kinematical conventions are shown in Fig. 7. Applying the diagrammatical rules given in [3], we find



**Fig. 7.** Kinematics relevant for the first diagram contributing to the radiative corrections of the leptonic decay width

$$\begin{aligned}
\mathcal{M}^{\mu\rho} = & -4\sqrt{\frac{m}{3}} \int \frac{d^3p_1}{2\varepsilon_{p_1}(2\pi)^3} \frac{d^3r_1}{2\varepsilon_{r_1}(2\pi)^3} \frac{d\tau}{\tau} \frac{d\tau'}{\tau'} \frac{d\tau_0}{\tau_0} \\
& \times \delta(p_2^2 - m^2) \delta(r_2^2 - m^2) \delta(q^2) \\
& \times \Theta(\omega \cdot p_2) \Theta(\omega \cdot r_2) \Theta(\omega \cdot q) (s - M^2) \\
& \times \text{Tr} \left[ \gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_2 + m) \right. \\
& \left. \times \phi^\rho (m - \not{r}_2) \gamma_\nu (m - \not{p}_2 + \not{\phi}') \right], \quad (40)
\end{aligned}$$

with  $s = (r_1 + r_2)^2$ . The integrations over  $\tau$ ,  $\tau'$  and  $\tau_0$  can be done using the delta functions induced by the on-mass shell condition of LFD. We also use momentum conservation at vertices, including spurion momenta, to express the various momenta. We have

$$\begin{aligned}
\int \frac{d\tau}{\tau} \delta(r_2^2 - m^2) &= \frac{1}{(1-x)(s-M^2)}, \\
\int \frac{d\tau'}{\tau'} \delta(p_2^2 - m^2) &= \frac{1}{(1-x')(s'-M^2)}, \\
\int \frac{d\tau_0}{\tau_0} \delta(q^2) &= \frac{1}{2\omega \cdot p(x-x')\tau_0},
\end{aligned}$$

with

$$\begin{aligned}
s' &= (p_1 + p_2)^2, \quad x = \frac{\omega \cdot r_1}{\omega \cdot p}, \quad x' = \frac{\omega \cdot p_1}{\omega \cdot p}, \\
\tau &= \frac{s - M^2}{2\omega \cdot p}, \quad \tau' = \frac{s' - M^2}{2\omega \cdot p}, \quad \tau_0 = \tau - \frac{m^2 - r_1 \cdot p_1}{\omega \cdot p(x-x')}.
\end{aligned}$$

Thus, we can express  $s$ ,  $s'$  and all other invariant quantities in terms of the integration variables. These integration variables can be conveniently chosen as  $x$ ,  $x'$ ,  $\vec{R}_T^2$ ,  $\vec{R}'_T^2$  and  $\vec{R}_T \cdot \vec{R}'_T$ , with

$$R = r_1 - xp \quad \text{and} \quad R' = p_1 - x'p, \quad (41)$$

and  $\vec{R}_T$  is the transverse part of  $R$  relative to the direction of  $\vec{\omega}$ , with the consequence that  $R^2 = -\vec{R}_T^2$  (see the appendices in [3] for more details). The calculation of the five-dimensional integral is done numerically using a Monte Carlo procedure.

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